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Classification of elementary particles by mass and spin or helicity.

Final degree project.

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Abstract

This work will describe and define the mathematical foundations and objects necessary for modeling the symmetries of Minkowski's spacetime, which give rise to Wigner's concept of elementary particles. After a brief introduction to differential geometry, group theory and representation theory, we will proceed to a more detailed study of Lie Groups in order to characterize their representations. Next, we will introduce the notion of the semidirect product and, in this way, provide structure to the Euclidean group and the Poincaré group so that we can identify the representations of the latter. Building on these concepts, we will finally be able to state and understand Mackey's Theorem for semidirect products, with which we will classify and define elementary particles as objects associated with the irreducible unitary representations of the Poincaré group.

Resum

En aquest treball, es descriuran i definiran els fonaments matemàtics i els objectes necessaris per modelar les simetries de l'espai-temps de Minkowski que donen lloc al concepte de partícules elementals de Wigner. D'aquesta manera, després d'una breu i concisa introducció a la geometria diferencial, la teoria de grups i la teoria de representacions, farem un pas endavant i ens introduirem més específicament en l'estudi dels Grups de Lie, per tal de caracteritzar les seues representacions. A continuació, entrarem en la definició del producte semidirecte i d'aquesta manera dotarem d'estructura al grup euclidià i al grup de Poincaré, de manera que podrem identificar les representacions d'aquest últim. Finalment podrem enunciar i entendre el Teorema de Mackey per a productes semidirectes, amb el qual classificarem i definirem les partícules elementals com a objectes associats a les representacions unitàries irreductibles del grup de Poincaré.

1 Introduction, goals and motivations.

It is well known that physics and mathematics have played an important role in shaping our understanding of the world. We could even say that it is impossible to imagine the world without these two disciplines. Throughout the course of history, people responsible for expanding the frontiers of mathematical and physical knowledge have been extremely valued for their indirect contributions to the improvement of our quality of life, in terms of the convenience that comes from knowing how the world and its plurality work when we interact with it and take advantage of it. However, it is well recognized that the idea of these two sciences has been subject to a very pronounced temporal evolution. It should even be emphasized that the relationship (or boundary) between the two of them has also been subject to obvious deformations.

But how is it that these two disciplines provide knowledge? Through which mechanisms do they interact with nature and with each other to identify structures and build *truth*? Although these two questions are very ambitious and have answers of high complexity, it is possible to make a first approximation of how they should be approached.

Reality is always observed and processed through the human lens; in other words, we only access and experience the world through our senses and organize such information with our language. To question ourselves «what is the true face of reality?» makes no sense when asked independently of human interaction with it: we can only talk about the world and how it works with a human mouth

and through a human brain. But now, in this context, what are mathematics and in what way are they linked to physics? It is often said that mathematics is a «language» that encodes the laws that «nature» follows. However, I would argue that such a definition is based on a misinterpretation. It is indeed some kind of language (although not universal), but it does not encode the laws that nature follows; rather, it encodes the way humans obtain information about reality through those underlying laws. This rectification emerges from what, in my opinion, is the main characteristic of the mathematical language: its ability to rigorously formalize and generalize our *physical* intuitions of reality (notice how the word «physics» has already appeared while defining mathematics through this fundamental trait). That is to say, we interact with matter and through that interaction we form physical intuitions about the world that allow us to identify patterns and discuss them with other human beings. However, while doing so, and in need of establishing a common background for us to communicate and share such intuitions to take advantage of them, we build tools that capture, in a rigorous and unequivocal way, this knowledge. Therefore, we obtain structures that allow us not only to establish connections among each other, but also to generalize such concepts and surpass our limitations on how far our physical interactions with reality can go. That is how mathematics is born.

The relationship between physics and mathematics arises as a consequence of the definition of the latter: physics, as a discipline, utilizes mathematical tools with regard to the intuitions they embody to obtain basic laws of interaction between the objects that constitute matter. On the other hand, mathematics can be considered a distinguished discipline because, once such basic structures are built to provide firm playground for physics to work, it is possible to create more and more objects and study their properties and connections with other pre-existing mathematical tools, building layers of complexity that may bear no connection to how humans experience nature (or, perhaps, it could be that some of those structures represent intuitions that are yet to be experienced), but that have other utilities.

Now, in regard to the first question raised, which mechanisms do these two disciplines use to provide knowledge? One of the most important ones is the construction of theories. These theories are usually based on a handful of assumed postulates that are later mathematized and that provide the first building block over which the theory is developed through deduction. However, it is a physicist's work to reduce such postulates in order to obtain a simpler description of reality, although still being able to model and explain a myriad of possible physical systems.

I believe this work to be an excellent example of how mathematics and physics interact and generate theories capable of simplifying our descriptions of reality, while simultaneously, through mathematical formalization of our physical intuition, gain epistemological access to phenomena we cannot even dream of directly experiencing: the subatomic particles. For this reason, the work will be divided into the following sections: firstly, in section 2, we will show how mathematics develops the structures needed to formalize our knowledge and intuitions around one of the most important concepts in modern physics: symmetries. We will see how these structures converge on Mackey's theorem in section 3.1, and we will use it to analyze the unfolding of the unitary irreducible representations of the Poincaré group by building a method derived from the theorem called «*Mackey's machine*» shown in section 3.2. To do the classification, we will first characterize the Poincaré group and its double universal covering, the spin group in sections 3.3 and 3.4. With this, we will be finally ready to perform the classification in section 3.5. It is at this point in which we will dive into the physical interpretation of said representations in section 4.1, where

the goal is to derive through well sustained arguments how this interpretation emerges naturally from the structure of the considered symmetries. Therefore, the practical result of this process is the classification of the elementary particles through the unitary irreducible representations of the Poincaré group; hence, the title of this work. Once the classification is finished, and in order to justify rigorously the correspondence between said representations and the elementary particles, in section 4.2 we will be able to see how the convergence of physics and mathematics depicted in this work is capable of simplifying our description of one of the most complete theories in modern physics by removing one of the principles used to derive it: the principle of correspondence in Quantum Field Theory. Finally, we will end this work with section 4.3 by providing a definition of «elementary particle» that justifies the whole classification and physical interpretation.

2 Description of mathematical tools used.

As mentioned above, this work consists of, firstly, the investigation of the essential mathematical objects that characterize and embody the human intuitions linked to the physical experiences with symmetries. These tools provide us the opportunity to describe them formally and unequivocally, and help us completely understand the link between those transformations and the most important and basic laws and relations in nature, hidden in the depths of physical knowledge. To that end, we will start by presenting three of the most important mathematical fields in terms of their contributions to modern physics: differential geometry, group theory and representation theory. These three disciplines will constitute the entire basis on which we will build our knowledge of the existence of the observed particles, and also of the not-observed ones.

With the objective of making the reading of this part of the work more cohesive and comprehensible, Figure 1 shows the summarized relation between the most important definitions presented in this mathematical introduction. Notice how every succession of definitions ultimately converges on Mackey's theorem, which shows that the mathematical tools introduced are the minimum necessary to build and understand such theorem.

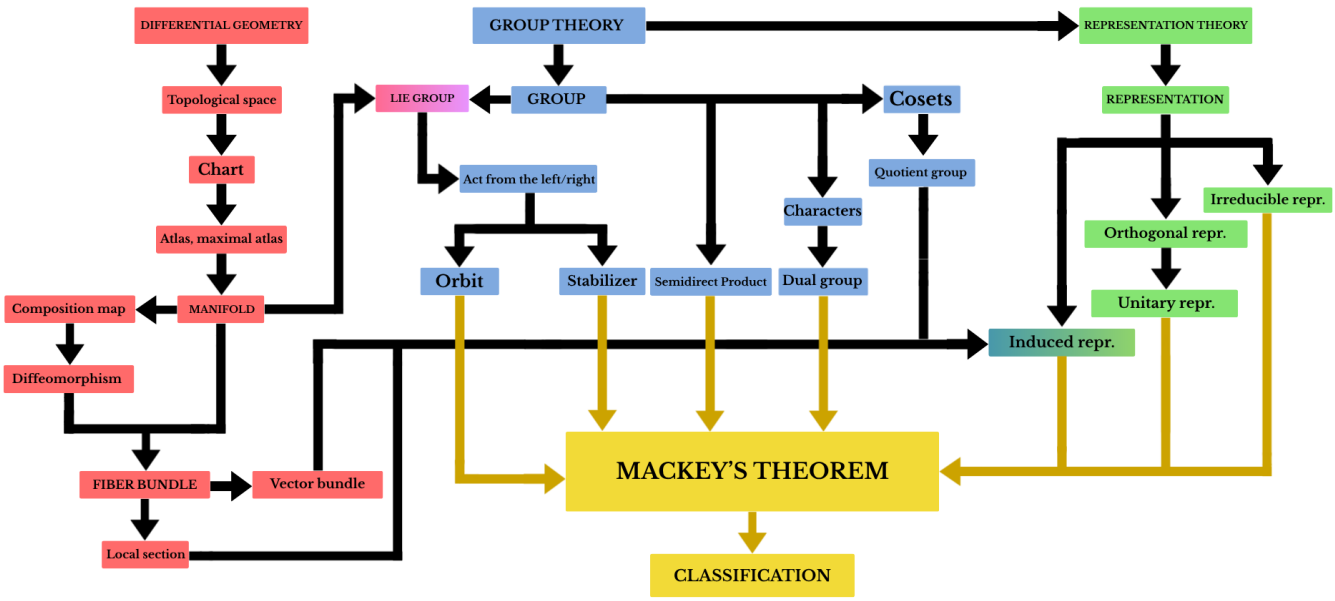


Figure 1: Definition diagram of the mathematical tools needed for Mackey's theorem construction.

In the following sections, we use «**Def.**» to begin a definition and «**.**» to mark its end.

2.1 Introduction to differential geometry.

Differential geometry is the mathematical discipline that studies the geometry of smooth spaces, and it combines techniques of differential calculus and linear algebra. To start working with it, we need to define its basic building block: *manifolds*. However, as we can see in the diagram 1, we will firstly introduce a series of concepts that will help us doing so, starting with the most basic definitions, that enable us to formally define concepts like limits, continuity and connectivity: topology and the topological space. The material presented in this mathematical introduction is mainly based on [1].

Def.1.1: Given a set of elements X, we can endow it with a *topology*. For doing so, we select a collection of subsets $\{\tau\}$ of the set X such that:

1. $\emptyset \in \{\tau\} \quad ; \quad X \in \{\tau\}$
2. The union of an arbitrary number of elements in $\{\tau\}$ is in $\{\tau\}$.
3. The intersection of a finite number of elements in $\{\tau\}$ is in $\{\tau\}$.

By doing so, we can say that the pair $(X, \{\tau\})$ is a *topological space***.** This being said, we can now fully enter the process of building the concept of manifold. For that, we will define what a chart is:

Def.1.2: Let $(X, \{\tau\})$ be a topological space and $\mathcal{O} \subset X$ an open subset. Then, given the following homeomorphism (i.e a bijective and continuous function between topological spaces that has a continuous inverse function):

$$\varphi : \mathcal{O} \to \mathcal{U} \subset \mathbb{R}^n$$

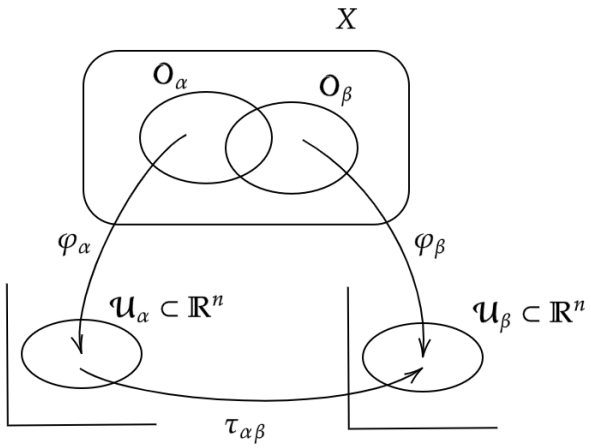
We can build the pair $(\mathcal{O}, \varphi)$, which is called a *chart*. It basically formalises the idea of assigning coordinates to each and every element of the open subset $\mathcal{O}$**.** Now, let us take the next step and establish what an atlas is:

Def.1.3: Let $\{\mathcal{O}_\alpha\}$ be an *open covering*, i.e a collection of open sets such that $\bigcup_\alpha \mathcal{O}_\alpha = X$, and $\{\varphi_\alpha\}$ a collection of charts assigned to each open set. Then, the set:

$$A = \{(\mathcal{O}_\alpha, \varphi_\alpha)\}_{\alpha \in I} \ ; \ \text{where } I \equiv \text{some set of indices}$$

is called an *atlas* of X. Intuitively, this enables us to assign coordinates to every point in X**.** However, we can bring ourselves into a position in which we have two charts $(\mathcal{O}_\alpha,\varphi_\alpha)$ and $(\mathcal{O}_\beta,\varphi_\beta)$, such that $\mathcal{O}_\alpha \cap \mathcal{O}_\beta \neq \emptyset$ (put with words, there are two possible sets of coordinates we can assign to these points in the intersection); we can then ask there to be a *transition map* between them defined as

$$\tau_{\alpha\beta} = \varphi_\beta \circ \varphi_\alpha^{-1} : \mathcal{U}_\alpha \subset \mathbb{R}^n \to \mathcal{U}_\beta \subset \mathbb{R}^n$$



This is what we informally call a *change of coordinates*. If we were to define all possible $\tau_{\alpha\beta}$ in such a way that they turned out to be class C^k (that is, it has continuous derivatives up to order k) or more, then we can talk about having a $C^k-Atlas$. And now, finally, if we supplement the atlas with every possible chart that is consistent with the other charts at the intersections (i.e the class of the transition maps of those new charts with all the pre-existing ones are C^k or more), then we will have constructed a maximal C^k-Atlas $\hat{A}$.

Now, after all this introductory definitions, we can at last determine what a manifold is, specifically a smooth manifold:

Def.1.4: Let $(X, \hat{A})$ be a pair of a topological space $\mathrm{X} \equiv (\mathrm{X}, \{\tau\})$ and a maximal C^k-Atlas $\hat{A}$ of X. Then we call this pair a C^k-*manifold*. If the charts assign coordinates belonging to $\mathbb{R}^n$, then the manifold is n-dimensional. If $k \to \infty$, we will have a *smooth manifold*.

This definition, which seems quite simple, enables us to speak of topological spaces in which we have induced a «C^k structure» through the atlas. It is, basically, the generalization of the concept of curve or surface to arbitrary dimensions. The charts that reside in the atlas allow us to deform open sets of X, locally, to open subsets of $\mathbb{R}^n$; in other words, locally, a manifold of $\dim = n$ resembles $\mathbb{R}^n$ as a topological space[1]. This makes of the manifold a fundamental object, being the building block of differential geometry.

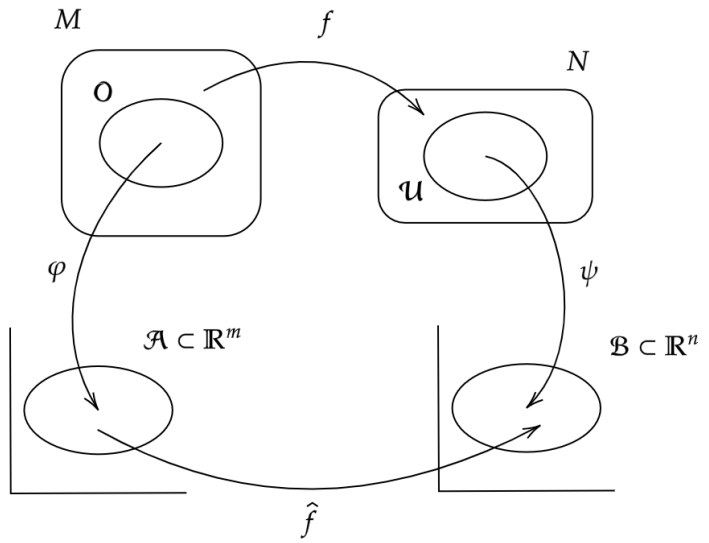
Once we have defined it, we can follow the diagram 1 to see that this mathematical tool will allow us to further define fiber bundles (which will be introduced in no time), but it will also give rise to a hybrid object, lie groups. This object, however, will be portrayed in the next section on group theory, corresponding to the blue-coloured part of the diagram.

Now, following the red-coloured part of the diagram, we set out to build some complexity over the manifold structure, defining another important object: fiber bundles. Nevertheless, we still need another previous, more simple definition about smooth maps on manifolds.

Def.1.5: Given two manifolds (M, $\hat{A}$) and (N, $\hat{B}$) of dimensions m and n respectively, and given a map $f : \mathrm{M} \to \mathrm{N}$, we can define the *composition map*:

$$\hat{f} \equiv \psi \circ f \circ \varphi^{-1} : \varphi(\mathcal{O}) \subset \mathbb{R}^m \to \psi(\mathcal{U}) \subset \mathbb{R}^n$$

[1]To be locally similar to $\mathbb{R}^n$ means that, although a manifold may have a complicated global structure, every point has a sufficiently small neighbourhood that can be identified with an open subset of Euclidean space $\mathbb{R}^n$. This property is provided by the system of charts defining the manifold structure. Informally, saying that a manifold «looks like $\mathbb{R}^n$ locally» is a more intuitive way of expressing the fact that each point admits a neighbourhood homeomorphic to an open subset of $\mathbb{R}^n$, highlighting how manifolds generalize familiar geometric spaces.



Where $(\mathcal{O},\varphi) \in \mathcal{A} \subset \mathbb{R}^m$; $(\mathcal{U},\psi) \in \mathcal{B} \subset \mathbb{R}^n$; $\mathcal{O} \subset$ M, U $\subset$ N, $f(\mathcal{O}) \subset \mathcal{U}$. We will say that f is of class C^k if every possible $\hat{f}$, for all possible pair of charts ψ and φ, is class C^k. If $k \to \infty$ then f is smooth. If, on top of that, m=n and f is bijective with both f and f^{-1} smooth, then we say f is a *diffeomorphism*, and M and N are *diffeomorphic* manifolds.

This being said, we can now properly define what a fiber bundle is.

Def.1.6: Let M be a manifold. To each point $x \in$ M we assign a manifold F_x such that every F_x is diffeomorphic to a common manifold F. We define and name:

1. M$\to$ Base.
2. $\mathrm{F}_x \to$ Fiber over the point x.
3. F$\to$ Typical fiber.
4. $B := \bigcup_{x\in \mathrm{M}} \mathrm{F}_x \to$ Total space (also a manifold).

We now define a surjective map called *canonical projection*, such that:

$$\pi : B \to \mathrm{M} \quad ; \quad \mathrm{F}_x \equiv \pi^{-1}(x)$$

So that to each point in B we assign the point $x \in$ M that is attached to the fiber F_x. The last item of the definition is the concept of *local product structure*: there exists a covering $\mathcal{O}_\alpha \subset$ M and a system of diffeomorphisms

$$\psi_\alpha : \pi^{-1}(\mathcal{O}_\alpha) \to \mathcal{O}_\alpha \times \mathrm{F}$$

where ψ_α is called *local trivialization*, such that for the map $\pi_1 : \mathrm{M} \times \mathrm{F} \to \mathrm{M}$ we have:

$$\pi_1 \circ \psi_\alpha = \pi$$

With all this requirements and objects, we finally define a fiber bundle as a 4-tuple (B, M, π, F). Alternatively, we can denote it as $\pi : B \to$ M if F is given by the context. If every fiber F_x of the bundle has an additional linear space structure, then we say we have a *vector bundle*. Once again, we can see on the diagram 1 that this object plays part on the construction of another hybrid, an even more complex one: *induced representations*. For such structures we will need a few more definitions, and therefore we will present them in the last section of the mathematical introduction, corresponding to the green-coloured part of the diagram.

Finally, we define:

Def.1.7: Let $\pi : B \to$ M be a fiber bundle. Then, we say that a smooth map such that

$$\sigma : \mathcal{O} \to B \mid \pi \circ \sigma = id_{\mathcal{O}} \text{ ; where } \mathcal{O} \subset \mathrm{M},\ id_{\mathcal{O}} \equiv \text{identity}$$

is a *local section* of the fiber bundle. If $\mathcal{O} \equiv \mathrm{M}$, we say it is a *global section*. We now have finished the red-coloured part of the diagram 1, containing the minimal knowledge we need to understand the geometrical structures that play a significant role in the classification of elementary particles through Mackey's Theorem.

2.2 Introduction to group theory: Lie groups.

We shift now our attention to another mathematical playground we introduced at the beginning of the work: group theory. This discipline is, basically, the one that studies the algebraic structures known as *groups*. To obtain the information related to the definitions that appear on this part of the work, we have mostly visited [1] and [2]. This section corresponds to the blue-coloured part of the diagram 1, and as we can see it all begins with said object:

Def.2.1: A group is an object that consists of a set X and a composition law $\circ$ that obey the following axioms:

1. $\forall a, b \in \mathrm{X}$, then $a \circ b \in \mathrm{X}$. In other words, $\circ$ is an *internal* composition law.
2. Associativity: $\forall a, b, c \in \mathrm{X}, (a \circ b) \circ c = a \circ (b \circ c)$.
3. Identity element: $\exists e \in \mathrm{X} \mid \forall a \in X, e \circ a = a \circ e = a$.
4. Inverse element: $\forall a \in \mathrm{X}, \exists a^{-1} \mid a^{-1} \circ a = a \circ a^{-1} = e$.

We call the pair $(\mathrm{X}, \circ)$ a group. This definition encapsulates and formalizes the idea of symmetry transformations, hence its importance in mathematics and theoretical physics is only a reflection of the connection between symmetries and the foundations of nature. In this way, groups are always identified as groups of transformation of something. In fact, for an arbitrary structure in a set X, (X, s), the automorphisms (bijective maps of an algebraic structure into itself that preserve operations defined in such structures) with the operation of composition of automorphisms are endowed naturally with the structure of a group. There is, however, as we said on the previous section, a specific type of hybrid that comes from the perspective of differential geometry, and that turns out to be of particular interest: Lie groups. Let us formally define them:

Def.2.2: A Lie group is a group that, in turn, has the structure of a manifold, and that it fulfills the condition that the maps

$$\rho : \mathrm{G} \times \mathrm{G} \to \mathrm{G} \quad ; \quad \rho : (g, h) \to gh^{-1}$$

are smooth maps. This last condition is equivalent to ask that the operations of multiplication and the transition to the inverse element are to be smooth maps. This ensures the compatibility between the two structures. This symbiosis of algebraic and differential-topological structures makes Lie groups very rich objects, as it allows them to be studied from both geometric and algebraic perspectives. As a consequence, several important structures naturally arise on Lie groups, such as left- and right-invariant vector fields and Lie algebras. These structures provide powerful tools for studying properties of Lie groups, as they encode infinitesimal information about the group and often simplify its analysis. However, the mentioned structures will not be discussed in the development of this work.

All the following definitions that are portrayed in the blue-coloured section of the diagram 1 are meant to build specific types of groups, each with its own particularities, and almost all of

them are objects that play a direct role in Mackey's theorem, as it is easy to visualize using the golden-coloured lines. Firstly, we will define a type of group that will be of the utmost importance, being that the Poincaré group belongs in this category: the semidirect product groups.

Def.2.3(1): Let G be a Group and G_A the group of the automorphisms of G. Now, let $G_{\tilde{A}}$ be a subgroup of G_A and $\Lambda(g) \in G$ the image of $g \in G$ by the automorphism $\Lambda \in G_{\tilde{A}}$. Then, we define the semidirect product $G \rtimes G_{\tilde{A}}$ as the set of the ordered pairs (g, Λ) with the product:

$$(g, \Lambda) \cdot (g', \Lambda') = (g\Lambda(g'), \Lambda\Lambda')$$

Which consequently defines:

$$\text{Identity element: } e = (e, I)$$
$$\text{Inverse element: } (g, \Lambda)^{-1} = (\Lambda^{-1}(g^{-1}), \Lambda^{-1}).$$

This is a very general definition. However, in this case we will use a more restricted one presented in [3], for practical reasons. Here is the alternative definition used to represent the Poincaré group, together with the specific restrictions that bring us to it:

Def.2.3(2): Let G be a Lie group with a closed Lie subgroup H and a normal Lie subgroup N (i.e $gng^{-1} \in G$; $\forall g \in G, \forall n \in N$). Then, G is the semidirect product $G = N \rtimes H$ if:

1. $\forall g \in G \rightarrow \exists n \in N$ and $\exists h \in H \mid g = (n, h) = nh$
2. $(n, h) \cdot (n', h') = (nhn'h^{-1}, hh')$

Here we can clearly identify the restriction made: we have established that N is a normal subgroup for us to make a correspondence between the subgroup $G_{\tilde{A}}$ and H, such that the explicit action of this automorphisms is made through the following expression: $\Lambda(n) = hnh^{-1}$. In other words, there is a homomorphism $\tau : H \rightarrow \text{Aut}(N)$ so that $\tau(h)n = hnh^{-1}$.

Once we have finally introduced the concept of a semidirect product, we will include some other important definitions related to groups and group theory that will help us work with Makey's theorem. For that, we will start looking at the groups as *groups of actions on a set* (although we shall go deeper on this terms in the next section regarding representation theory). Before jumping on it, we must point out that the following definition is not the most general one regarding actions of groups on sets, as we will specifically talk about the actions of *Lie groups* on *manifolds*. However, for the purpose of this work, it will suffice.

Def.2.4: Let G be a Lie Group and let M be a manifold. We will say that G *acts from the left* on M if for each group element $g \in G$ we assign a diffeomorphism L_g that satisfies:

$$L_g : M \rightarrow M \quad ; \quad L_{gh} = L_g \circ L_h \quad ; \quad L_e = id_M \quad ; \quad \forall g, h \in G$$

We can also define this type of action but from the right:

$$R_g : M \rightarrow M \quad ; \quad R_{gh} = R_h \circ R_g \quad ; \quad R_e = id_M \quad ; \quad \forall g, h \in G$$

We will then say that G *acts from the right* on M. A shorthand notation usually used is to represent this actions is:

$$x \rightarrow L_g x =: gx \quad ; \quad x \rightarrow R_g x =: xg \quad ; \quad \forall x \in M, g \in G$$

Although we need to always keep in mind that the element of the group g does not actually act on its own on x; in other words, «gx» does not represent a product of any kind, it is just a shorthand notation for «$\mathrm{L}_g x$».

The importance of this type of actions goes beyond their ability to act on the manifold. Other very interesting structures arise from the relation between these two objects: orbits and stabilizers. Let us define them both, as they have a protagonist role in Makey's Theorem playground.

Def.2.5: Given a Lie Group G that acts from the left on a manifold M, we can define the *orbit* $\mathcal{O}_x \in \mathrm{M}$ *of the point* $x \in \mathrm{M}$ as a subset in M that sprouts from the restriction

$$\mathcal{O}_x := \{y \in \mathrm{M} \mid \exists g \in \mathrm{G} \text{ such that } y = gx\}.$$

Def.2.6: Given a Lie Group G that acts from the left on a manifold M, we can define the *stabilizer of a point* $x \in \mathrm{M}$ (also named the *isotropy group* or the *little group*) as the subgroup $\mathrm{G}_x \subset \mathrm{G}$ such that:

$$\mathrm{G}_x = \{g \in \mathrm{G} \mid gx = x\}$$

Alternatively said, it is the subgroup made from the elements of the group that leave the point x fixed. With this new pair of definitions, there are only two concepts left to be established. This last concepts from group theory will help us understand the meaning of the *induced representation*, which will be presented at the end of the introduction to representation theory, and that is a fundamental piece in Makey's Theorem.

Def.2.7: Let H be a subgroup of G. Given an element $g \in \mathrm{G}$, we define the *left cosets* of H in G as the sets obtained by:

$$g\mathrm{H} = \{g' \in \mathrm{G} \mid g' = gh \ ; \ h \in \mathrm{H}\} \subset \mathrm{G}$$

In other words, we create subsets called left cosets for each $g \in \mathrm{G}$ acting from the left to every element of H. This definition can be also made with *right cosets*, the only difference being that the element g is acting from the right. With this new object, we can further define another:

Def.2.8: Let N be a normal subgroup of G. Let us then consider the set of all left cosets, denoted as G/N. Being that N is a normal subgroup, we are entitled to define an operation between elements of G/N such that:

$$a\mathrm{N} \circ b\mathrm{N} = (ab)\mathrm{N} \ ; \ a, b \in \mathrm{G}$$

With this operation, it is straight forward to see that the set of all left cosets is endowed with the structure of a group. We will name this group the *quotient group*.

Now, we finally define one last and quite simple concept that will enable us to work with Mackey's theorem.

Def.2.9: Let G be a be an abelian locally compact group. Then, a *character* is a continuous function:

$$\hat{f} : \mathrm{G} \to \mathbb{C} \mid |\hat{f}(g)| = 1, \hat{f}(g_1 \circ g_2) = \hat{f}(g_1)\hat{f}(g_2)$$

Where $\circ$ is the intern composition law. In other words, we can define it as if the image of the function is on S^1, the circle of radius 1. If we take into account the set of all characters of the

group G and equip it with the binary operation of the pointwise multiplication we then have a group, the *dual group* $\hat{\mathrm{G}}$, which is also abelian . We will see later, when we introduce the concept of unitary representations, that characters are endowed naturally with the structure of one-dimensional continuous unitary representations of G.

This being said, we have established all needed definitions corresponding to the blue-coloured section of the diagram 1 and consequently of group theory. We can then finally move on to the last mathematical introduction.

2.3 Introduction to representation theory.

Let us begin the immersion to the last branch of mathematics that we will introduce: representation theory. It is strongly related to group theory, as it describes a way in which groups act over linear spaces. As pointed out previously, groups are the formalisation of our intuitions linked to physical symmetries. As such, they can always be seen as groups of *transformations* of something, through their actions on a set (endowed with additional structures). In other words, there exists a rule with which we can associate a transformation to each element of a group, framed on a particular set.

This is where the concept of representation emerges, as they are a specific, very important, kind of actions: they act linearly on linear spaces. In other words, representations emerge from the place in which linearity and symmetries meet. Therefore, its importance is directly inherited from the distinguished position that linearity occupies within the scope of mathematics. Analogously to the previous mathematical introduction, this green-coloured part of the diagram 1 only contains the concept of representation and specific types of such object. The definitions corresponding to this section have been also obtained from [1] and [2]. Now, without further ado, let us formally define them:

Def.3.1: Let V be a linear space (i.e vector space) in which we have the intention to apply a group of transformations of symmetry G. Then, a homomorphism such that:

$$\rho : G \to \mathrm{Aut(V)} \equiv \mathrm{GL(V)} \quad ; \quad \rho(g\tilde{g}) = \rho(g)\rho(\tilde{g})$$

[2]is called a *representation of the group G in the vector space V*, (V,ρ)**.** In other words, to utilize the group as a group of transformations of a linear space, we need to ensure the compatibility of the linear structure with the symmetry operations and assign a transformation to every group element simultaneously. For that to happen, we force the correspondence between each element of the group and a linear operator through a map that reproduces the behaviour of the elements of the group; in other words, a homomorphism. Notice that for it to be a homomorphism, the image of ρ must be inside the automorphisms of the vector space V, not the endomorphisms, as, unlike Aut(V), this last one does not come endowed naturally with the structure of a group, for they do not necessarily have an inverse. The dimension of the space V is called the dimension of the representation ρ.

However, for us to work with Makey's Theorem, we will need to define yet a special kind of representations, the irreductible representations and the unitary representations.

[2]GL(V) denotes the *general lineal group* of the linear space V, which is another way of referring to the group of automorphisms of said linear space

Def.3.2: Given a representation (V, ρ) of the group G, if $\exists W \subset V \mid \rho(G)W \subset W\ ;\ W \neq \{\emptyset, V\}$, i.e there is a non-trivial invariant subspace, then we say the representation is *reducible*. On the contrary, if the only invariant subspaces that exist are the trivial ones, then we have an *irreducible representation*. In other words, if we have a non-trivial subspace which is simultaneously invariant under any operator given by the representation, then we can say there is another representation «hidden», a *subrepresentation*.

Now, we move on to the unitary representations. To define them, we firstly need to coin a concept known as $\rho-$*invariant scalar product*.

Def.3.3: Let (V, ρ) be the representation of a group G. If the linear space V is endowed with a scalar product h and the following rule

$$h(\rho(g)v, \rho(g)w) = h(v, w)\ ;\ \forall v, w \in V, g \in G$$

is fulfilled, then we say we have an *invariant scalar product*, and we also have a *compatible triple* (V, h, ρ). The representation in this compatible triple is called an *orthogonal representation*. If, furthermore, the linear space V is a complex one and the product is an hermitian inner product, then we have a *unitary representation*. With these basic definitions, we will introduce a more complex concept, the hybrid we have previously mentioned in group theory. This object, as we can see in the diagram 1, emerges from the intersection between the three mathematical disciplines we have introduced, and has a protagonist role in Mackey's theorem: *induced representations*. For practical purposes, and to avoid the usage of measure theory, this following definition has been obtained from [3], as well as the diagram that accompanies the explanation.

Def.3.4: Let G be a group and $H \subset G$ a closed subgroup. Let us consider then that $\rho : H \to \mathrm{Aut}(\mathcal{H})$ a unitary[3] representation of the group H on the Hilbert space $\mathcal{H}$. With these objects, we can define the product $G \times \mathcal{H}$ and the action of G over it:

$$G \times (G \times \mathcal{H}) \to G \times \mathcal{H}\ :\ (g, (g', v)) \to (gg', v)\ ;\ \forall g, g' \in G; v \in \mathcal{H}$$

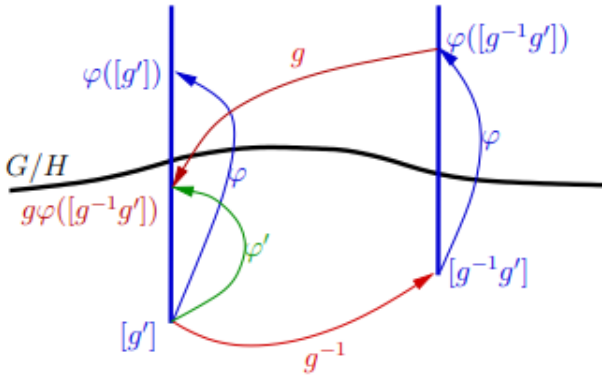
And, furthermore, we define the following *equivalence relation*:

$$(g, v) \backsim (g', v') = (gh, \rho(h^{-1})v)\ ;\ \forall g, g' \in G; v, v' \in \mathcal{H}; h \in H$$

The *equivalence class* of (g, v), denoted as $[g, v]$, is formed by every pair $(gh, \rho(h^{-1})v)$ with every possible $h \in H$. Notice, however, that these definitions have two important consequences: on the one hand, we can observe that the equivalence relation is invariant under the action of G, which enables us to define the action of G *over* the equivalence class $g \cdot [g', v] := [gg', v]$. On the other hand, it is important to note that the equivalence classes previously defined can be seen as a set, and we can also see that the set comes endowed naturally with the structure of a vector bundle, in which the base is formed out of the quotient set G/H (notice that the first part of every element of each equivalent class is nothing else than a left coset) and the fiber is the Hilbert space $\mathcal{H}$ ($\pi : G/H \times_H \mathcal{H} \to G/H$)[4].

[3] A general definition of an induced representation would not include the specification of «unitary», but in this work we present such particular case of induced representation directly, for it is the one used in the classification of elementary particles.

[4] This notation «$\times_H$» is used to distinguish it from the trivial bundle.



We can also point out that H becomes the *structure group* of the fiber (we will not discuss this in detail here due to space constraints). Now, after all that, let us consider the linear space of global sections of $\mathrm{G/H}\times_H\mathcal{H} =: E$, $\Gamma(E)$. Notice that the sections, by their definition, act on the elements of G/H, which we will write as $[g]$. Then, we will define the *representation of* G *induced by* ρ, denoted as $R_\rho = \mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}}(\rho) : G \to \mathrm{Aut}(\Gamma(E))$ that acts as follows:

$$(R_\rho(g)\varphi)([g']) \equiv \varphi'([g']) = g\varphi([g^{-1}g']) \ ; \ \varphi \in \Gamma(E); g, g' \in \mathrm{G}$$

So that $\varphi' = R_\rho(g)\varphi$ is another section of the vector bundle.

We have finally gone through each and every definition depicted in the diagram 1, and as we can see, all that remains is to finally connect and sew the final products through the golden arrows onto Mackey's theorem.

3 Classification of particles by mass and spin or helicity.

After this long but necessary mathematical introduction, portraying three of the most important fields in this discipline that all happen to converge elegantly in Mackey's Theorem, we have all the necessary tools to examine and understand such theorem, whose consequences are of the utmost importance for the purpose of this work. For that reason, we now move onto the core of the theorem, which we will then be applied on the soon to be discussed Poincaré Group. As we will see, the results this theorem develops can be used to build a «method machine». This method gives us a way of systematically obtain the classification of all the irreducible unitary representations of a regular semidirect product.

3.1 Mackey's Theorem on semidirect products.

Let us first state the theorem. The following construction follows from the Mackey analysis of unitary representations of semidirect products with abelian normal subgroup (see [4] and [5]). However, the explicit form of the theorem that we present here has been obtained from [6]:

Theorem (Mackey): Let $\mathrm{G} = \mathrm{N} \rtimes \mathrm{H}$ be a semidirect product, where N is a normal and abelian subgroup of G and H is a closed subgroup of G. N, H and G are all Lie groups. Then, let $\mathcal{O}_\chi$ be an orbit of H in the dual group $\hat{\mathrm{N}}$ through the point $\chi \in \hat{\mathrm{N}}$. Now, Let us pick a point χ_0 in $\mathcal{O}_\chi$ and let H_{χ_0} the stabilizer of χ_0, $\mathrm{H}_{\chi_0} \subset H$. Let ρ be a unitary irreducible representation of H_{χ_0}, and let $R_\rho \equiv \mathrm{Ind}_{\mathrm{H}_{\chi_0}}^{\mathrm{H}}(\rho)$ the representation of H induced by ρ. Let us define now a unitary operator $U(n)$ on Γ(E), the linear space of sections of the vector bundle $E \equiv G/H \times_H \mathcal{H}$, where n is any element of N. This operator is defined by:

$$(U(n)\sigma)(h) = (h \cdot \chi_0)(n)\sigma(h) \ ; \ \sigma \in \Gamma(\mathrm{E}), h \in \mathrm{H}, n \in \mathrm{N} \tag{1}$$

We finally define:

$$L(nh) = U(n)R_\rho(h)\ ;\ n \in \mathrm{N}, h \in \mathrm{H} \tag{2}$$

Where *L is a unitary irreducible representation of G*. Moreover, given that G is a regular[5] semidirect product, *every* possible unitary irreducible representation of G can be constructed like this. If another $\chi_0 \in \mathcal{O}$ is chosen, then the result will be an representation equivalent to the first one.

Now, we first need to point out a very important characteristic of the theorem that will help us understand its link to the classification of the elementary particles. It seems each collection of equivalent unitary irreducible representations of G correspond uniquely to a specific orbit of H in the dual group $\hat{\mathrm{N}}$ *and* to the unitary irreducible representation of the stabilizer of a point inside that orbit (each point will give us an equivalent representation of the collection).

This very powerful theorem enables us to build every unitary irreducible representation of a regular semidirect product, and given that the Poincaré group is such an object, it can be applied to it. Later on, when we do, we will be able to interpret and trace each equivalent collection as an elementary particle of a certain type with an orbit of the dual group of the normal subgroup of the transformations of Poincaré.

3.2 Classification method.

Before entering the final stage of the work, let us first summarize the theorem characteristics that I have mentioned above within a handful of straightforward instructions. It is no coincidence that Mackey's theorem is sometimes called «*Mackey's machine*», for it delivers a method to compute and classify every unitary irreducible representation of a semidirect product. In this section we follow the description of the method found in [2], which consists of the following steps:

1. Determine the set $\hat{\mathrm{N}}$ of all characters of N.
2. Classify all orbits $\mathcal{O}$ in $\hat{\mathrm{N}}$ under the subgroup H.
3. Select an element χ_0 in a given orbit $\mathcal{O}$ and determine the stabilizer $\mathrm{H}_{\chi_0} \subset \mathrm{H}$.
4. Take an irreducible representation ρ of H_{χ_0} and form the induced irreducible unitary representation $R_\rho \equiv \mathrm{Ind}_{\mathrm{H}_{\chi_0}}^{\mathrm{H}}(\rho)$.
5. Finally, form the unitary irreducible representation L of the group G with eq. (1) and (2).

With this easy to follow method, we will now land all this necessary mathematical knowledge into our physical playground: the Poincaré group of transformations.

However, in this case, given the extension limit of the work, we will not be explicitly deriving the irreducible representations of the stabilizers, and neither will we form the corresponding induced representations or the following unitary irreducible representations of the Poincaré group (that is, we will not explicitly go through the development of steps 4 and 5). Alternatively, we will dedicate our time to understanding and visualizing the connection between those representations and the elementary systems and elementary particles.

[5] To be regular is a property linked to *measure theory*, a branch of mathematics that we have not included in this work, given that it is possible to grasp the vast majority of the formalities needed to understand the classification without entering such branch. Nevertheless, for a wider, more rigorous study on the classification of the unitary irreducible representations of a semidirect product, one could introduce said mathematical structures.

3.3 The Poincaré Group.

Let us characterize the Poincaré Group as it is portrayed in [2] and [7], where we can find a more detailed exposition. In special relativity, the structure of space-time is established through the *Minkowski space-time*, a pseudo-Euclidian space of signature $(1,3)$ (or $(3,1)$), which is a trivial flat pseudo-Riemannian manifold. In other words, it is a finite-dimensional real affine 4-space equipped with a non-degenerate, symmetric bilinear form on the tangent space at each point of spacetime, which we will call Minkowski inner product, with metric signature either $(1,3)$ or $(3,1)$. It is defined through the Minkowski metric tensor:

$$\eta = \text{diag}(-1,1,1,1) \;\text{ or }\; \eta = \text{diag}(1,-1,-1,-1) \tag{3}$$

This structure, combined with the second postulate of special relativity and the homogeneity of spacetime and isotropy of space allows us to define the *spacetime interval* between two arbitrary events as:

$$(\Delta s)^2 = c^2(t_1 - t_2)^2 - (x_1 - x_2)^2 - (y_1 - y_2)^2 - (z_1 - z_2)^2 \tag{4}$$

Then, we are capable to define the Poincaré group as the one that is formed by all the isometry transformations in the Minkowski space. Equally said, the transformations keep the «distance» between events, the spacetime interval, invariant. It can also be said (it is equivalent) that this kind of transformations leave the scalar product defined by Minkowski's metric invariant. This group turns out to be incredibly important, for it also can be read as the group of all possible transformations between inertial frames of reference, in which the laws of physics must be the same, following the second postulate of special relativity.

We have then established that the Poincaré group is the full symmetry of special relativity, and it contains translations in time and space and Lorentz transformations. Its mathematical structure, derived from the previous definition is of a ten-dimensional non-compact Lie group. We can study its form by noticing that the Lorentz transformations form the *orthogonal group* $\mathrm{O}(1,3) = \{L \in \mathrm{GL}(4,\mathbb{R}) \mid L^{\mathrm{T}}\eta L = \eta\}$. It can be easily derived from this equation that $\det L = \pm 1$. In terms of this condition and the sign of L^0_0, we can divide Lorentz transformations in four disconnected pieces:

1. $\mathrm{L}^{\uparrow}_{+}$: $\det(\mathrm{L}) = +1$, $\mathrm{sign}\mathrm{L}^0_0 = +1$
2. $\mathrm{L}^{\uparrow}_{-}$: $\det(\mathrm{L}) = -1$, $\mathrm{sign}\mathrm{L}^0_0 = +1$
3. $\mathrm{L}^{\downarrow}_{+}$: $\det(\mathrm{L}) = +1$, $\mathrm{sign}\mathrm{L}^0_0 = -1$
4. $\mathrm{L}^{\downarrow}_{-}$: $\det(\mathrm{L}) = -1$, $\mathrm{sign}\mathrm{L}^0_0 = -1$

Moreover, only the transformations $\mathrm{L} \in \mathrm{L}^{\uparrow}_{+}$ form a subgroup called the *proper orthochronus Lorentz group*, which is the connected component of the identity (i.e, it consists of all Lorentz transformations that can be reached from the identity in a continuous manner). This is the group of transformations that we will take into account for the definition of the Poincaré group in this work.

We can identify the four-dimensional abelian group of spacetime translations as a normal group, and the proper orthochronus Lorentz group as a closed subgroup. Hence, the Poincaré group can also be depicted as the following semidirect product:

$$\mathbf{P} = \mathbb{R}^{1,3} \rtimes \mathrm{SO}^{+}(1,3) \equiv \mathrm{N} \rtimes \mathrm{L} \tag{5}$$

With the following inner product defined:

$$(n_1, L_1) \cdot (n_2, L_2) = (n_1 + L_1 n_2, L_1\ L_2) \tag{6}$$

Where $\mathrm{SO}^+(1,3)$ is the *simple orthogonal group* (i.e $\mathrm{SO}^+(1,3) = \{L \in \mathrm{GL}(4,\mathbb{R}) \mid L^T \eta L = \eta,\ \det(L) = 1,\ L^0_0 > 0\}$). We have referred to $\mathrm{SO}^+(1,3) \equiv \mathrm{L}^\uparrow_+$ also as L, as we will do for the rest of the work. Moving on, in quantum field theory, the Poincaré group comes in the form of its universal cover[6]

$$\mathbb{R}^{1,3} \rtimes \mathrm{SL}(2,\mathbb{C}) \tag{7}$$

Where $\mathrm{SL}(2,\mathbb{C})$ is the *special lineal group* of degree 2 over the complex numbers (i.e $\mathrm{SL}(2,\mathbb{C}) = \{\Lambda \in \mathrm{GL}(2,\mathbb{C}) \mid \det(\Lambda) = 1\}$). Unlike the representations of $\mathrm{SO}^+(1,3)$, this type of group is able to describe fields with spin $1/2$, as we will shortly show.

3.4 The Spin Group.

As we said earlier, we will not consider the first but the second definition of the Poincaré group, featuring the $\mathrm{SL}(2,\mathbb{C})$ group, for it will provide us of a more profound physical understanding of the representations obtained via Mackey's theorem. But, how is it possible to portray the Poincaré group in that way? Well, not only it is possible, but it also turns out to be the *universal covering* of this group. Let us show why by following the analysis depicted in [8] and [2].

To do so, we will establish a connection between $\mathrm{L}^\uparrow_+$ and the $\mathrm{SL}(2,\mathbb{C})$ of all 2×2 complex matrices of determinant one. Firstly, with each four-vector $x^\mu \in \mathbb{R}^{1,3}$ one can associate a 2×2-hermitian matrix X by the formula:

$$X = \sum_{\mu=0}^{3} x^\mu \sigma_\mu = \begin{bmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{bmatrix}$$

Where σ_μ is the set of four hermitian matrices of the form:

$$\sigma_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad \sigma_0 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Also known as the Pauli matrices ($\sigma_{1,2,3}$) and the identity (σ_0). This map gives us a one-to-one correspondence between a four-vector and a hermitian 2×2 matrix. It is easy to demonstrate that:

$$\det(X) = x^\mu x_\mu$$

This will be useful shortly. For now, let us write:

$$\hat{X} = \Lambda X \Lambda^\dagger\ ;\ \Lambda \in \mathrm{SL}(2,\mathbb{C})$$

It is trivial to see that $\hat{X}$ is also hermitian, and therefore we can obtain its correspondent vector $\hat{x}$. We can visualize this equation as a transformation given by the element Λ; there is, thus, a map $\lambda : \Lambda \to L_\Lambda$ that acts on the elements $\Lambda \in \mathrm{SL}(2,\mathbb{C})$ to give us transformations between hermitian

[6] A universal cover of a Lie group is built with a covering map (which is also a homomorphism) $\chi : \tilde{X} \to X$ that acts on the double universal cover onto the original Lie group such that the universal cover $\tilde{X}$ is simply connected.

matrices, ergo vectors in $\mathbb{R}^{1,3}$, which we will denote as L_Λ. Moreover, and here lies the importance of such a map, we can easily see that the equation that defines the transformation implies that, given that $\det(\Lambda) = 1$:

$$\hat{x}^\mu \hat{x}_\mu = \det(\hat{X}) = \det(X) = x^\mu x_\mu$$

Hence, the transformation L_Λ leaves the scalar product invariant and, consequently, represents a Lorentz transformation. This means $\det(L_\Lambda) = +1$ or -1. However, it is possible to see that the only real possibility is for it to be $+1$. This is due to the fact that if both possibilities were true, then the set of all transformations L_Λ would be disconnected, topologically. Nevertheless, given that we know $\mathrm{SL}(2,\mathbb{C})$ is connected and the map $\lambda : \Lambda \to L_\Lambda$ is continuous, this is impossible. Thus, we can say the only possibility is for it to be $\det(L_\Lambda) = +1$ (we take $+1$ because we need the Identity to be in the set for it to be a subgroup, and $\det(\mathbb{I}) = +1$). Consequently, the map $\lambda : \Lambda \to L_\Lambda$ is a *homomorphism* of $\mathrm{SL}(2,\mathbb{C})$ into $\mathrm{L}_+^\uparrow$, which allows us to portray, as previously said, the Poincaré group in terms of $\mathrm{SL}(2,\mathbb{C})$. In fact, given that $\mathrm{SL}(2,\mathbb{C})$ is simply connected, it is the universal covering group of the proper Lorentz group and, consequently, of the Poincaré group when used in a semidirect product with $\mathbb{R}^{1,3}$.

One interesting property of this matrices that will be of use later is that:

$$\Lambda = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \to \sigma_2 \Lambda (\sigma_2)^{-1} = \begin{bmatrix} \delta & -\gamma \\ -\beta & \alpha \end{bmatrix} = (\Lambda^T)^{-1} \tag{8}$$

3.5 Classification.

Let us finally get into the part of the work that gives it its name. The classification was originally obtained by Wigner [9]. In what follows, we will review it following the exposition of Barut [2]. To classify the elementary particles we will use the previously detailed method obtained from Mackey's theorem on the unitary irreducible representations of the Poincaré group, and we will show how they come indexed with two variables that can be interpreted as mass and spin. Let us go step by step, firstly identifying our semidirect product:

$$\mathcal{P} = \mathbb{R}^{1,3} \rtimes \mathrm{SL}(2,\mathbb{C}) \equiv \mathrm{N} \rtimes \mathrm{S} \tag{9}$$

Defined by the following composition law:

$$(n_1, \Lambda_1) \cdot (n_2, \Lambda_2) = (n_1 + L_{\Lambda_1} n_2, \Lambda_1 \ \Lambda_2) \tag{10}$$

Where the action of $\mathrm{SL}(2,\mathbb{C})$ on N is through L_Λ.

Then, we apply the «Mackey machine» method:

1. **Determine the set $\hat{\mathrm{N}}$ of all characters of N:** we identify N with the four-dimensional abelian group of spacetime translations. This group has the usual sum as its intern composition law. Because of that, to define the set of all characters we need the continuous functions which are subject to the following restrictions:

$$\hat{f}(n) \in \mathbb{C}$$
$$|\hat{f}(n)| = 1$$
$$\hat{f}(n_1 + n_2) = \hat{f}(n_1)\hat{f}(n_2)$$

It is quite clear that this set of functions might be related to the exponential with imaginary argument, for it covers each and every condition. However, knowing that n is a four-dimensional vector and that we need a scalar in the argument of the exponential function, we state that all the possible characters must be of the form:

$$\hat{f}_p(n) = e^{ip_\mu n^\mu} \tag{11}$$

Where we see the set of characters is thus built around a parameter that determines each function, the 4-dimensional covector p_μ. It is also possible to note that the dual group $\hat{\mathrm{N}}$ is then isomorphic to N, which can be seen as the space where p_μ live. This way, we can identify the dual group $\hat{\mathrm{N}}$ as the isomorphic copy N of the set of characters given by the isomorphism (11). In other words, to every $p_\mu = (p_0, p_1, p_2, p_3) \in \hat{\mathrm{N}}$ of the dual group taken into consideration there corresponds a character $e^{ip_\mu n^\mu}$. We will later give the (yet to be obvious) physical interpretation to the elements of this dual group.

2. **Classify all orbits $\mathcal{O}$ in $\hat{\mathbf{N}}$ under the subgroup S:** let us now see how the group $\mathrm{SL}(2,\mathbb{C})$ acts on $\hat{\mathrm{N}}$, so that we can determine the orbits generated by it. The transformation must be applied through the action of the homomorphic group L_Λ on the four-vectors $n \in \mathrm{N}$. This being said, we apply such transformations to the characters, to see in which way the transformation affects the elements of our dual group:

$$\hat{f}_{p'}(n) = \hat{f}_p(n') = \hat{f}_p(L_\Lambda^{-1} n) \equiv \hat{f}_p((L_\Lambda^{-1})^\mu{}_\nu n^\nu) = \exp\{ip_\mu (L_\Lambda^{-1})^\mu{}_\nu n^\nu\} = \exp\{ip^\rho \eta_{\rho\mu}(L_\Lambda^{-1})^\mu{}_\nu n^\nu\}$$

Using the fact that $L^{\mathrm{T}}\eta L = \eta \to L^{\mathrm{T}}\eta = \eta L^{-1} \to (L_\Lambda^{\mathrm{T}})_\rho{}^\mu \eta_{\mu\nu} = \eta_{\rho\mu}(L_\Lambda^{-1})^\mu{}_\nu$, we can write:

$$\exp\{ip^\rho \eta_{\rho\mu}(L_\Lambda^{-1})^\mu{}_\nu n^\nu\} = \exp\{ip^\rho (L_\Lambda^{\mathrm{T}})_\rho{}^\mu \eta_{\mu\nu} n^\nu\} = \exp\{ip^\rho (L_\Lambda)^\mu{}_\rho n_\mu\} = \exp\{i(L_\Lambda p)^\mu n_\mu\}$$

Hence, our set of elements of the dual group transform in the following way:

$$\hat{f}_{p'}(n) \equiv \hat{f}_{L_\Lambda p}(n) \quad ; \quad p \to L_\Lambda p$$

Which shows that the group $\mathrm{SL}(2,\mathbb{C})$ acts in the same manner on N and $\hat{\mathrm{N}}$. Moving on to the orbits, we can choose a point $p_q \in \hat{\mathrm{N}}$ and the corresponding orbit $\mathcal{O}_{p_q}$ is formed by the set of all $L_\Lambda p_q, \forall L_\Lambda \in L_+^\uparrow$. This structure has an important consequence: the product $p^2 = p^\mu p_\mu$ is invariant under any L_Λ, and therefore *it is constant on every orbit* (if we take p_q and another point of its orbit, $L_\Lambda p_q$, we can calculate both its squared products and find out that $(L_\Lambda p_q)^2 = (L_\Lambda)^\mu{}_\nu (p_q)^\nu (L_\Lambda)_\mu{}^\nu (p_q)_\nu = p_q^2$). With this result, we can use such invariant to distinguish each and every orbit, for each will have assigned a different p^2. The way the group $\mathrm{SL}(2,\mathbb{C})$ acts on $\hat{\mathrm{N}}$ enables us tu speak in this terms and use this invariant as a label.

Finally, we can state that every orbit will be contained in one of the hyperboloides of the form:

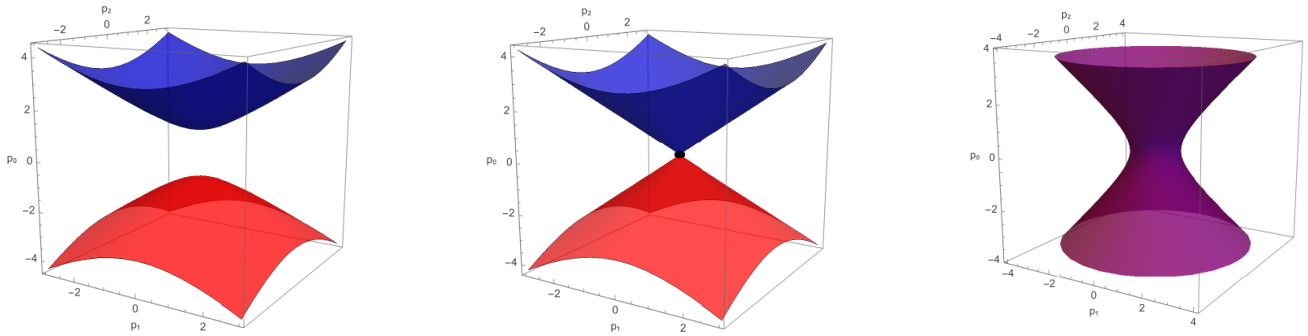
$$p^2 = (p_0)^2 - (p_1)^2 - (p_2)^2 - (p_3)^2 = m^2 \; ; \; m \in \mathbb{R} \tag{12}$$

We can now identify various cases depending on the value m. If we have $m^2 > 0$, we then have a two-sheeted hyperboloid, one in the upper side of the cone and one in the lower side, corresponding to the sign of the first element of the four-vectors, $p_0 > 0$ for the upper one and $p_0 < 0$ for the lower one. Then, we have the possibility of $m^2 = 0$, which describes a cone.

However, we can distinguish between 3 types of orbits here, because the orbit assigned to the element $p^\mu = 0$ is only itself ($L_\Lambda p^\mu = 0$; $\forall L_\Lambda$). This way, we have an orbit corresponding to this sole point, and furthermore, similarly to the case $m^2 > 0$, we have the cone facing upwards corresponding to $p_0 > 0$ and the cone facing downwards corresponding to $p_0 < 0$ (obviously both of them without the $p^\mu = 0$ point). Finally, we also have the case in which $m^2 < 0$. Unlike the other two cases, this one creates an equation that describes a one-sheeted hyperboloid, it only has one orbit associated. We can see the three-dimensional graphics that represent this orbits in Figure 2.

To sum up, we have six types of orbits:

(a) $\mathcal{O}_m^+ : (p_0)^2 - (p_1)^2 - (p_2)^2 - (p_3)^2 = m^2 \; ; \; m > 0, p_0 > 0$
(b) $\mathcal{O}_m^- : (p_0)^2 - (p_1)^2 - (p_2)^2 - (p_3)^2 = m^2 \; ; \; m > 0, p_0 < 0$
(c) $\mathcal{O}_{im} : (p_0)^2 - (p_1)^2 - (p_2)^2 - (p_3)^2 = -m^2 \; ; \; m > 0$
(d) $\mathcal{O}_0^+ : (p_0)^2 - (p_1)^2 - (p_2)^2 - (p_3)^2 = 0 \; ; \; m = 0, p_0 > 0$
(e) $\mathcal{O}_0^- : (p_0)^2 - (p_1)^2 - (p_2)^2 - (p_3)^2 = 0 \; ; \; m = 0, p_0 < 0$
(f) $\mathcal{O}_0^0$: the point $p_\mu = 0 = (0,0,0,0) \; ; \; m = 0.$



$m > 0$ $\quad$ $m = 0$ $\quad$ $m^2 < 0$

Figure 2: Classification of the orbits in $\hat{N}$ under the action of S.

The hyperboloides corresponding to $p_0 > 0$ are coloured in blue, while the ones with $p_0 < 0$ are in red. For $m = 0$ we have also depicted the point of the orbit (f) with a small black sphere.

3. **Select an element χ_0 in a given orbit $\mathcal{O}$ and determine the stabilizer $\mathbf{S}_{\chi_0} \subset \mathbf{S}$:** let us do so systematically with every one of the six orbits.

   (a) $\mathcal{O}_m^+$: we can choose the element $p_a = (m, 0, 0, 0)$ as a representative of the orbit, with $m > 0$. Let us remember that the stabilizer of the point p_a is the subgroup of S that, when applied on the point, it leaves it invariant. First, we start by calculating the 2×2-hermitian matrix corresponding to that four-vector:

$$X_1 = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} = mI$$

Now, we see which elements $\Lambda \in \mathrm{SL}(2,\mathbb{C})$ make that $\hat{X} = \Lambda X \Lambda^* = X$. It is trivial to see that

$$\hat{X}_1 = \Lambda X_1 \Lambda^\dagger = m\Lambda\Lambda^\dagger = X_1 = mI \rightarrow \Lambda\Lambda^\dagger = I$$

And so, we can identify that the stability subgroup S_a of the point p_a is the unitary group SU(2).

(b) $\mathcal{O}_m^-$: choosing the element $p_b = (-m,0,0,0)$ as the representative, we can repeat the same process and it is obvious that we will trivially find, once again, that the stability subgroup S_b of the point p_b is the unitary group SU(2).

(c) $\mathcal{O}_{im}$: for this orbit, we will choose the point $p_c = (0,m,0,0)$, whose corresponding 2×2-hermitian matrix is:

$$X_3 = \begin{bmatrix} 0 & -m \\ m & 0 \end{bmatrix} = mi\sigma_2$$

Thus, the condition the stability group must follow is:

$$\sigma_2 = \Lambda\sigma_2\Lambda^\dagger \rightarrow (\sigma_2)^{-1}\Lambda\sigma_2 = (\Lambda^\dagger)^{-1} \tag{13}$$

Having said this, given that $(\sigma_2)^{-1} = \sigma_2$, equations (8) and (13) imply that

$$(\Lambda^T)^{-1} = (\Lambda^\dagger)^{-1} \rightarrow \Lambda = \bar{\Lambda}$$

Consequently, the stability group S_c of the point p_c is the group of 2×2-real unimodular matrices, $\mathrm{SL}(2,\mathbb{R})$.

(d) $\mathcal{O}_0^+$: we choose the point $p_d = (\frac{1}{2},0,0,\frac{1}{2})$. The corresponding matrix is:

$$X_4 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

The condition ends up being:

$$\Lambda X_4 \Lambda^\dagger = X_4 \rightarrow \Lambda = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \rightarrow \begin{bmatrix} \alpha\bar{\alpha} & \alpha\bar{\gamma} \\ \bar{\alpha}\gamma & \gamma\bar{\gamma} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

With this equation we determine that $\gamma = 0$, $\alpha = e^{i\theta}; \theta \in [0,2\pi]$, $\beta = z \in \mathbb{C}$ and, being that $\det(\Lambda) = 1$, we end up having $\delta = e^{-i\theta}$. Finally, the stability group of the point p_4 consists of all the 2×2 matrices of the form:

$$\begin{bmatrix} \exp(i\theta) & z \\ 0 & \exp(-i\theta) \end{bmatrix} \in \mathrm{S}_d$$

We will show now that this group S_d is isomorphic to a semidirect product itself in a very simple way. Let us write an arbitrary element $k_i \in \mathrm{S}_d$ and the product $k_1 \cdot k_2$:

$$k_i = \begin{bmatrix} \exp(i\frac{\theta_i}{2}) & \exp(-i\frac{\theta_i}{2})z_i \\ 0 & \exp(-i\frac{\theta_i}{2}) \end{bmatrix} \rightarrow k_1 k_2 = \begin{bmatrix} \exp(i\frac{\theta_1+\theta_2}{2}) & \exp(-i\frac{\theta_1+\theta_2}{2})(z_1 + \exp(i\theta_1)z_2) \\ 0 & \exp(-i\frac{\theta_1+\theta_2}{2}) \end{bmatrix} \tag{14}$$

This way, if we set $k_i = (z_i, \theta_1)$, we immediately obtain the following composition law:

$$k_1 k_2 = (z_1, \theta_1)(z_2, \theta_2) = (z_1 + \exp(i\theta_1) z_2, \theta_1 + \theta_2) \tag{15}$$

Which ends up showing that it can actually be concieved as the semidirect product $\mathrm{S}_d = \mathbb{R}^2 \rtimes S^1$ between the two-dimensional translation group $\mathbb{R}^2 = \{(z, 0)\}$ and the rotation group $S^1 = \{(0, \theta)\}$. Given that the next step of the method is to take the irreducible representations of each stabilizer, then we will apply the same method for this stabilizer S_d to obtain its unitary irreducible representations.

(e) $\mathcal{O}_0^-$: this case is very similar to the previous one. Choosing $p_e = (-\frac{1}{2}, 0, 0, \frac{1}{2})$, we have:

$$X_5 = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \to \Lambda X_5 \Lambda^\dagger = X_5 \to \begin{bmatrix} -\beta\bar{\beta} & -\beta\bar{\delta} \\ -\delta\bar{\beta} & -\delta\bar{\delta} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \tag{16}$$

We then end up having $\beta = 0$, $\gamma = z \in \mathbb{C}$, $\delta = e^{i\theta}$; $\theta \in [0, 2\pi]$ and, given that $\det(\Lambda) = 1$, we also have $\alpha = e^{-i\theta}$. Finally, the stabilizer S_5 group is formed by:

$$\begin{bmatrix} \exp(-i\theta) & 0 \\ z & \exp(i\theta) \end{bmatrix} \in \mathrm{S}_e \tag{17}$$

Which shows that $\mathrm{S}_e \approx \mathrm{S}_d$. Such relation between these groups allows us to use the same procedure for the following steps of the method, hence we will not be seeing both of them, we will only extend the results correspondent to the orbits $\mathcal{O}_0^+$ to this case.

(f) $\mathcal{O}_0^0$: finally, we get to the last of the six types of orbits, and also the simplest one. In this case, being that the orbit only consists of one point and such point is $p_f = (0, 0, 0, 0)$, the stability group consists, trivially, of the whole $\mathrm{SL}(2, \mathbb{C})$ group itself.

4. Take an irreducible representation ρ of $\mathbf{H}_{\chi_0}$ and form the induced irreducible unitary representation $R_\rho \equiv \mathbf{Ind}_{\mathbf{H}_{\chi_0}}^{\mathbf{H}}(\rho)$. Finally, form the unitary irreducible representation L of the group G with formulas (1) and (2): let us do so systematically with each one of the six orbits again.

(a) $\mathcal{O}_m^+$; SU(2): The group SU(2) has unitary irreducible representations ρ^j of dimension $2j + 1$; $j = 0, \frac{1}{2}, 1, \frac{3}{2}, ...$ The corresponding induced representations of the Poincaré group will therefore be labelled with two parameters: m and j. We denote them as $R^{m,+;j}$.

(b) $\mathcal{O}_m^-$; SU(2): for this other orbit, we have the same stabilizer. We will then proceed exactly like the other orbit above, resulting in the induced representations labeled by m and j. This time, we denote them as $R^{m,-;j}$.

(c) $\mathcal{O}_{im}$; $\mathrm{SL}(2, \mathbb{R})$: for the $\mathrm{SL}(2, \mathbb{R})$ group we can identify three series of unitary irreducible representations. Firstly, we have the principal series $\rho^{i\sigma,\varepsilon}$, where $\sigma \in \mathbb{R}$ and $\varepsilon = 0$ or 1. Then we have the discrete series ρ^n, with $n = 0, 1, 2, ...$ These discrete series can be of two types, $\rho^{n,+}$ and $\rho^{n,-}$. Finally, we have the supplementary series, denoted ρ^λ, with $-1 < \lambda < 1$, $\lambda \neq 0$. These series of representation give rise to the irreducible unitary representations $R^{im;i\sigma,\varepsilon}$, $R^{im;n,\pm}$ and $R^{im;\lambda}$ of the Poincaré group.

(d) $\mathcal{O}_0^+$; S_d: as we have previously said, this stabilizer is, in turn, a semidirect product. That's why, in order to obtain their unitary irreducible representations, we would apply Mackey's theorem method on $S_d = \mathbb{R}^2 \rtimes S^1$. After doing so and obtaining the induced unitary irreducible representations, we obtain the following pair of series: $R^{0,+;h}$, where $h = 0, \pm\frac{1}{2}, \pm 1, ...$ and $R^{0,+;r,\varepsilon}$, where $r > 0$ and $\varepsilon = 0, 1$.

(e) $\mathcal{O}_0^-$; S_e: this case is completely analogous to the previous one, in a way that we finally obtain the series of unitary irreducible representations written as $R^{0,-;h}$, where $h = 0, \pm\frac{1}{2}, \pm 1, ...$ and $R^{0,-;r,\varepsilon}$, where $r > 0$ and $\varepsilon = 0, 1$.

(f) $\mathcal{O}_0^0$; $\mathrm{SL}(2,\mathbb{C})$: in this last case, the $\mathrm{SL}(2,\mathbb{C})$ group has a principal and a supplementary series of unitary irreducible representations. Representations of the principal series are labelled by two numbers: $i\lambda$, with $\lambda \geq 0$ and $j = 0, \frac{1}{2}, 1, \frac{3}{2}, ...$ On the other hand, the supplementary ones have only one single real number as a label, λ, where $-1 < \lambda < 1$, $\lambda \neq 0$. With this, we finally denote as $R^{0,0;i\lambda,j}$ and $R^{0,0;\lambda}$ the corresponding unitary irreducible representations of the Poincaré group.

To sum up, the final classification of all the unitary irreducible representations of the Poincaré group is:

(a) $R^{m,+;j}$, $m > 0$, $p_0 > 0$, $j = 0, \frac{1}{2}, 1...$

(b) $R^{m,-;j}$, $m > 0$, $p_0 < 0$, $j = 0, \frac{1}{2}, 1...$

(c)
i. $R^{im;i\sigma,\varepsilon}$, $m > 0$, $\sigma \geq 0$, $\varepsilon = 0, 1$
ii. $R^{im;n,\pm}$, $m > 0$, $n = 0, 1, 2...$
iii. $R^{im;\lambda}$, $m > 0$, $-1 < \lambda < 1$, $\lambda \neq 0$

(d)
i. $R^{0,+;h}$, $m = 0$, $p_0 > 0$, $h = 0, \pm\frac{1}{2}, \pm 1...$
ii. $R^{0,+;r,\varepsilon}$, $m = 0$, $p_0 > 0$, $r > 0$, $\varepsilon = 0, 1$

(e)
i. $R^{0,-;h}$, $m = 0$, $p_0 < 0$, $j = 0, \pm\frac{1}{2}, \pm 1...$
ii. $R^{0,-;r,\varepsilon}$, $m = 0$, $p_0 < 0$, $r > 0$, $\varepsilon = 0, 1$

(f)
i. $R^{0,0;i\lambda,j}$, $m = 0$, $p_0 = 0$, $\lambda \geq 0$, $j = 0, \pm\frac{1}{2}, \pm 1...$
ii. $R^{0,0;\lambda}$, $m = 0$, $p_0 = 0$, $-1 < \lambda < 1$, $\lambda \neq 0$

Where we can see the association with the orbits (a)-(f); the first set of upper index characterizes de orbit and the second one characterizes the unitary irreducible representations of the stability subgroup of such orbit.

4 Nature's fundamental constituents: elementary particles.

Once we have finally completed the whole classification of the unitary irreducible representations of the Poincaré group, let us move on to understand the meaning of this representations and the physical interpretation that derives from them, as developed by [9]. In order to further support this interpretation, we also follow [10] in discussing the role of the structure of space-time and its symmetries in quantum mechanics.

4.1 Physical interpretation of the classification.

Firstly, we will take a look at the group of space-time translations and a unitary representation of this group on a Hilbert space $\mathcal{H}$: the translation operator. This representation works as follows:

$$\rho : \mathbb{R}^{1,3} \to \text{Aut}(\mathcal{H}) \; ; \; \rho(a_\mu) = e^{-\frac{ia_\mu P^\mu}{\hbar}} \tag{18}$$

Here, P^μ is the generator of such symmetries, the four-momentum operator. This representation is reducible, for there is a subgroup of the well defined moment states on which the translation operator acts as:

$$e^{-\frac{ia_\mu P^\mu}{\hbar}} |p^\mu\rangle = e^{-\frac{ia_\mu p^\mu}{\hbar}} |p^\mu\rangle \tag{19}$$

And the result, given that it is multiplied by a phase, belongs to the same subgroup. However, we can see that this is also an eigenvalue equation, in which the eigenvalues of the translation operator are, as we can see, of a very familiar form: they can be seen as the characters of the translation group (with the sole exception of the constant $\hbar$, which we can take as $\hbar = 1$ given an appropriate unit system) which, in turn, are continuous unitary irreducible representations. Given this relation between the translation group and the characters, it is possible to physically identify the parameter used to label the characters: the four-momentum. In other words, the dual group of the translation group consists of the set of four-momentum covectors.

With this realization, the physical understanding of the unitary irreducible representations of the Poincaré group and their labels comes naturally to us. Let us follow the method's lead and identify the other parameters and their implications. Hence, the next step turns out to be the classification of all orbits in the dual group under the subgroup SL(2,$\mathbb{C}$). We previously arrived to the conclusion that every orbit was labelled by the invariant p^2 in a way that each one will be contained in the following hyperboloides:

$$p^2 = (p_0)^2 - (p_1)^2 - (p_2)^2 - (p_3)^2 = m^2 \; ; \; m \in \mathbb{R} \tag{20}$$

However, knowing that p_μ is the four-momentum covector, the physical meaning of the m parameter comes clear: the mass. The reason is simple: it is well known from the definition of the four-momentum that its squared value is a Lorentz invariant that adopts the form:

$$p^2 = m^2 u^2 = m^2 c^2 \tag{21}$$

Where m is the mass and u^μ the four-velocity. It is possible now to do the obvious comparison, and through the same process as the last identification, argue that with the proper unit system in which $c = 1$ the m parameter clearly corresponds to the mass. We can now begin to ask ourselves what physical object is the yielder of this mass and four-momentum. The answer, as possibly anticipated by the work's title, is an elementary particle. We will give a solid explanation that reinforces such correspondence later, although we will be using the term and explaining the following aspects of the physical interpretation of the parameters with such object in mind for storytelling purposes. For now, let us give physical interpretation to the parameters that label the representations.

$\mathcal{O}_m^+$ and $\mathcal{O}_m^-$: For the $\mathcal{O}_m^+$ orbit, it is stated that $m > 0$ and $p_0 > 0$. We can see the mass is a positive value. However, the interesting aspect of the classification here is the second condition, which shows that the first component of the four-momentum is positive. Such component is

nothing less than the total relativistic energy. Particles represented by this orbit are then *ordinary* particles that form the ordinary matter.

Secondly, we have the $\mathcal{O}_m^-$ orbit, characterized by $m > 0$ and $p_0 < 0$. It is then possible to identify such orbits with *antiparticles*, given that their total energy is negative-valued and following Feynman–Stückelberg's interpretation [11].

For both of this orbits, their corresponding representations are labelled by another parameter j, such that the dimension of the linear space in which the automorphisms built by the representation act is $2j+1$; $j = 0, \frac{1}{2}, 1, \frac{3}{2}, ...$ It is quite obvious then to see that such parameter represents *spin*, given that the stats that represent a particle with spin j must have $2j+1$ components corresponding to each projection of the third component of spin.

$\mathcal{O}_{im}$: After these two orbits, we get to the $\mathcal{O}_{im}$ orbit. This one is characterized by $m^2 < 0$, which means that we have imaginary mass. This has a lot of important implications, as clearly shown in [12]: if we force E to be real-valued and we take Einstein's formula for the total energy of a particle, we then get to:

$$E = \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}} \in \mathbb{R} \ ; \ \text{if } m \in \mathbb{C} \rightarrow 1 - \frac{v^2}{c^2} < 0 \rightarrow v > c \tag{22}$$

This kind of particles have therefore a *superluminal* velocity, and are called *tachyons*. Even though this results come naturally in this classification, this kind of particles have never been observed, and for very good reason: if tachyons could be detected it would mean that they can interact somehow with matter. However, given that they travel faster than the speed of light, this would lead to (at least for now) unavoidable paradoxes caused by the violation of causality. Therefore, given that such particles are not to be observed, we won't be giving any further explanation to the other parameters that label their correspondent representations.

$\mathcal{O}_0^+$ and $\mathcal{O}_0^-$: Continuing with the next orbits, we have $\mathcal{O}_0^+$ and $\mathcal{O}_0^-$, that are distinguished for having $m = 0$ and $p_0 > 0$ or $p_0 < 0$ respectively. This type of particles, due to being massless, travel at the speed of light. The first orbit of this massless particles corresponds to the ones with positive energy, the ordinary massless particles. Observed particles of such kind are *photons*, while *gluons* have been inferred rather than observed, and are predicted in theory to be massless. On the other hand, we have massless particles with negative energy that, analogously to the case of the particles with positive mass, means that they are massless antiparticles. Now, we have two series of representations for each of these orbits, which are labelled by the parameters $h = 0, \pm\frac{1}{2}, \pm 1, ...$ for the first kind of representations and $r > 0$, $\varepsilon = 0, 1$ from the second one. Let us first focus on h. We previously established that the stability group of the orbit $\mathcal{O}_0^+$ was the semidirect product $\mathrm{S}_4 = \mathbb{R}^2 \rtimes S^1$ between the two-dimensional translation group $\mathbb{R}^2 = \{(z, 0)\}$ and the rotation group $S^1 = \{(0, \theta)\}$. To obtain the irreducible unitary representations of such stabilizer would mean to apply once again Mackey's theorem. When doing so, we obtain that the orbits are determined by the action of S^1 on the dual group of $\mathbb{R}^2$, which is completely analogous to the case of $\mathbb{R}^{1,3}$. Therefore, this action can be viewed explicitly as:

$$(0, \theta)(\hat{z}, 0)(0, -\theta) = (e^{i\theta}\hat{z}, 0) \tag{23}$$

Where we have used the notation introduced in (14) and (15). The result is simply a rotation of angle θ. The orbits will then be, according to its definition, circles in the complex plane $\hat{\mathbb{R}}^2$,

with the center at (0,0) and radius $r \geq 0$. This way, it is possible to distinguish between the orbit with $r = 0$ and the ones with $r > 0$. For the first kind of orbits, the stability group is the whole group of rotations S^1, and its unitary irreducible representations are one-dimensional: $e^{ih\theta}$, where $h \in \mathbb{Z}/2$; their induced representations give rise to $R^{0,+;h}$. Let us now introduce the physical object that will give us the physical interpretation of this parameter: helicity. Its definition is simply the projection of the total angular momentum over the movement direction [10]:

$$\hat{h} = \frac{\vec{J}\vec{p}}{|\vec{p}|} \tag{24}$$

Notice that, for rotations around the propagation axis, the states of a massless particle satisfy:

$$R(\theta)\,|p,h\rangle = e^{i\frac{\vec{J}\vec{p}}{\hbar|\vec{p}|}\theta}\,|p,h\rangle = e^{ih\theta}\,|p,h\rangle \;\;;\;\; h \in \mathbb{Z}/2 \tag{25}$$

And through a similar argument than the one used to identify the parameter p as the four-momentum we can state that, given that the generator of such rotations is the operator helicity and that the momentum states of massless particles are eigenstates of such rotations with eigenvalues that have the same form as the unitary irreducible representations of S^1, we clearly identify h with the helicity. This explains the difference between this number, that acquires negative values, while j can only be positive. On the other hand, we still have another set of representations corresponding to $r > 0$ and $\varepsilon = 0, 1$, for which we will not do such an extended analysis, for these representations feature particles with so-called *continuous spin*, and have not been observed, although they have been object of some interesting theoretical investigations [13]. For the orbit $\mathcal{O}_0^-$ the procedure is completely analogous, and we arrive to the same conclusions. Before moving on to the final orbit, we should mention that the photon is physically understood by the massless representations with helicities $h = \pm 1$. Since the photon is massless, no rest frame exists and only the two helicity states survive as physical degrees of freedom. These correspond to the two possible polarization states of the electromagnetic field. Furthermore, the photon is its own antiparticle, and so the negative-energy branch does not give rise to a distinct particle species. Therefore, the physical photon is associated with the massless positive-energy representations of helicities $h = \pm 1$, such that it is represented by the direct sum $R^{0,+;+1} \oplus R^{0,+;-1}$.

$\mathcal{O}_0^0$: Finally, we now pay attention to the orbit that consists of only one point, $\mathcal{O}_0^0$. This orbit has no energy and, therefore, no mass or momentum, for what the only possible physical interpretation is the obvious one: this «particle» corresponds to the non-existing non-being one, the vacuum. This orbit, as previously explained, gives rise to representations induced from unitary irreducible representations of $\mathrm{SL}(2,\mathbb{C})$. Among them, only the trivial representation, the one that associates the identity to each element of the group, admits a natural interpretation as the vacuum state. One way of understanding this is by noticing that the vacuum state cannot be influenced by a transformation such as translations, rotations or boosts, every observer whose frame of reference is connected by such transformations must see the same state, the vacuum:

$$\rho(g)\,|0\rangle = |0\rangle \;\;;\;\; g \in \mathbf{P} \to \rho(g) \equiv \mathbb{I}$$

The remaining representations that belong to the principal and supplementary series of $\mathrm{SL}(2,\mathbb{C})$, as stated by [6], have no established physical interpretation.

This being said, we can now summarize all this information on the classification of elementary particles through the association to the unitary irreducible representations of the Poincaré group, which is condensed in the following table:

$R^{m,+;j}$	Particles with $m > 0$ and spin j
$R^{m,-;j}$	Antiparticles with $m > 0$ and spin j
$R^{im;i\sigma,\varepsilon}$ $R^{im;n,\pm}$ $R^{im;\lambda}$	Tachyonic particles (not observed, non-existing)
$R^{0,+;h}$	Particles with $m = 0$ and helicity h
$R^{0,-;h}$	Antiparticles with $m = 0$ and helicity h
Trivial representation	The vacuum

Table 1: Classification of the elementary particles with the corresponding unitary irreducible representations of the Poincaré group.

4.2 The principle of correspondence.

We have finally completed the classification of the elementary particles by associating them to the unitary irreducible representations of the Poincaré group. Nevertheless, as previously stated, we have not justified this association, which at first could surprise us given it is not trivial nor intuitive in any way, at least not at first glance. Let us try to justify it by deriving an important result as done in [6], although adapted to this work by avoiding the usage of measure theory.

We focus on the unitary irreducible representations $R^{m,+;j}$, with $m > 0$, $p_0 > 0$, and $j = 0, \frac{1}{2}, 1....$ Now, let us work with the representation that features $j = 0$, of dimension $2j + 1 = 1$, meaning that the Hilbert space in which the image of the representation acts has dim$= 1$. We also know that the orbit that corresponds to such representations is:

$$\mathcal{O}_m^+ = \{p \in (\mathbb{R}^{1,3})^* \equiv \mathbb{P} \mid p^2 = m^2,\, p_0 > 0\}$$

And, therefore, the induced representation of the representations of the stability group of such orbit must act on the following vector space:

$$\mathcal{H} = L^2(\mathcal{O}_m^+)$$

Where the vectors are square integrable functions f such that:

$$f : \mathcal{O}_m^+ \to \mathbb{C}$$

For that reason, given that it is defined over the orbit, every function will surely follow the equation:

$$(p^2 - m^2)f(p) = 0 \tag{26}$$

We want now to express this equation in the spacetime frame, and for that we will perform the Fourier transform:

$$\psi(x) = \int_{\mathcal{O}_m^+} f(p)e^{-ixp}dp$$

And finally, in order to use the result from equation (26), we apply the operator $(\Box + m^2)$ on the Fourier transform:

$$(\Box + m^2)\psi(x) = (\Box + m^2)\int_{\mathcal{O}_m^+} f(p)e^{-ixp}dp = \int_{\mathcal{O}_m^+} (-p^2 + m^2)f(p)e^{-ixp}dp = 0$$

We therefore obtain:

$$\boxed{(\Box + m^2)\psi(x) = 0}$$

This shows that equation (26) was in fact the Fourier transform of a very familiar expression: the Klein-Gordon equation. This shows that the Hilbert space over which the Poincaré group acts through the unitary irreducible representation $R^{m,+;j=0}$ is formed by the physical states that correspond to a scalar particle of positive mass, as we set out to show.

Quantum Field Theory is usually introduced through the principle of correspondence, with which we transform a classical equation to obtain its quantum analogue by assigning to each observable its corresponding quantum operator, and applying such equation on a wave function. With this principle, the Klein-Gordon equation can be obtained from the relativistic equation of $p^\mu p_\mu = m^2$.

In this work, however, we have derived the explicit form of the Klein-Gordon equation based purely on the convergence between two fundamental restrictions: firstly, quantum mechanics show that physical systems, and in particular elementary particles, must be represented as quantum states by rays in a Hilbert space. Therefore, when we want a symmetry transformation to act over a physical state, we need it to be through a unitary representation of the group. On the other hand, special relativity dictates that the symmetry group of spacetime transformations must be the Poincaré group. When accepting both of these restrictions, and through Mackey's theorem, it is possible to directly obtain the wave equations satisfied by the states that represent elementary particles (here we have only derived the Klein-Gordon case, for it is the most simple one)[7]. We have done so without appealing to the principle of correspondence, but as a consequence of more fundamental principles.

4.3 The definition of elementary particle.

There is, however, another clarification to be made. In what sense are these particles elementary? The term does not refer to the impossibility of further subdivision into more fundamental constituents. Rather, it refers to the fact that their state space is transformed by symmetries through an irreducible unitary representation of the Poincaré group. Indeed, if a representation $R(g)$ on a Hilbert space $\mathcal{H}$ were reducible, then there would exist a non-trivial closed invariant subspace $K \subset \mathcal{H}$ such that $R(g)K \subset K$ for every group element g. The action of the symmetry group would never mix states belonging to the two sectors. For example, if we had $\mathcal{H} = \mathcal{H}_{m,j=0} \oplus \mathcal{H}_{m,j=1}$, a Poincaré transformation would never connect these two subspaces. From the point of view of the symmetry, the system would therefore split into independent components. An elementary particle, in Wigner's sense [9], is precisely a system for which no such decomposition exists.

This definition captures the essence of our intuitions with respect to an elementary system: it is not possible for two inertial observers whose frame of reference is connected through a trans-

[7] Dirac's equation and others are also possible to be derived in this way, as done in [6], but it is considerably more complicated (even, not known) for higher spin particles.

formation of the Poincaré group to observe two kinds of distinct particles with different mass and spin/helicity. That is what we mean by elementary: there are certain intrinsic properties (that actually label, as shown, the unitary irreducible representations of the Poincaré group) that characterize their identification, which cannot change under different inertial frames of reference.

Consequently, a composed object such as a proton may still correspond to an irreducible representation of the Poincaré group when regarded as a single-particle state. Irreducibility characterizes the action of spacetime symmetries on the state space, not the microscopic structure of the object.

5 Conclusions.

To sum up, we can say that we have achieved what we first set out to do at the beginning of the work: after a precise and adjusted mathematical introduction with the minimal content necessary to understand Mackey's theorem, we have applied that theorem in the form of a method called «Mackey's machine» in order to obtain the classification of the unitary irreducible representations of the Poincaré group. We have then moved on to the physical interpretation of that classification, where we have grounded the mathematical objects and parameters onto the physical plane, connected to our intuitions with respect to our experience with matter. In this process, we have relied on several important concepts: firstly, the structure of spacetime and the connection between translations and the four-momentum. Then, we have identified the mass as an invariant associated with the four-momentum, and we have proved its direct relation to the orbits of the dual group of $\mathbb{R}^{1,3}$ under the action of $\mathrm{SL}(2,\mathbb{C})$. To continue, we have systematically provided an interpretation for each parameter that labels the unitary irreducible representations of the representations with real-valued mass, with the exception of the representations corresponding to the point orbit, for which we have argued that the only representation with an established physical interpretation was the trivial one. In doing so, we have identified the spin j for $m > 0$ orbits and the helicity h for massless representations.

Finally, we have shown how the Klein-Gordon equation is derived solely from the unitary irreducible representation $R^{m,+;j=0}$ providing an argument that such representations act on the Hilbert spaces containing free elementary particles. Hence, it is possible to make a direct association between the unitary irreducible representations of the Poincaré group and the elementary particles, classifying them in accordance with the intrinsic properties of mass, spin and helicity. To make such a statement, however, we have had to define what an «elementary» particle is, providing a more complete understanding of the meaning of the word «irreducible».

If we had delved into other mathematical fields such as measure theory or used other objects such as systems of imprimitivity or Lie algebras and Casimir operators, our description would have been even more rigorous and complete and would have given us more tools to provide a more solid physical interpretation.

Nevertheless, this more modest approximation to the problem suffices to give form to the majority of the important ideas regarding the classification and its consequences. As stated in the project's introduction, this work provides a perfect example of how mathematics and physics are developed to obtain knowledge about our way of interacting with reality and matter. On the one hand, it is possible to observe in the mathematical introduction how this science builds its objects and treats them, formalizing our human physical intuitions of, for example, symmetries in

the form of groups and their representations. On the other hand, we have seen the exact moment in which physics takes the lead and sets out to do what it does best: provide a direct correlation between said mathematical objects and our experience of nature, thus building knowledge about reality through logical arguments. It is through this interplay that we have been able to build and unify theories gaining simplicity in their set of principles and complexity in their development and their ability to explain and predict. This work provides a clear example of such simplification of principles by removing the principle of correspondence, as explicitly shown.

This work therefore presents an approach to physics not usually encountered in a Physics Degree, one in which the connection with mathematics acquires explicit relevance in the understanding of the physical interpretation of such mathematical tools. It is often argued that physicists should avoid getting «lost» in mathematical and philosophical questions, whatever that may mean. However, in my view, this is an act of self-sabotage: the most exceptional physical breakthroughs happen when the convergence of these disciplines is acknowledged and nurtured.

6 Bibliography.

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